GABARITO DA QUARTA PROVA DE ALGEBRA II

Primeiro semestre de 2006

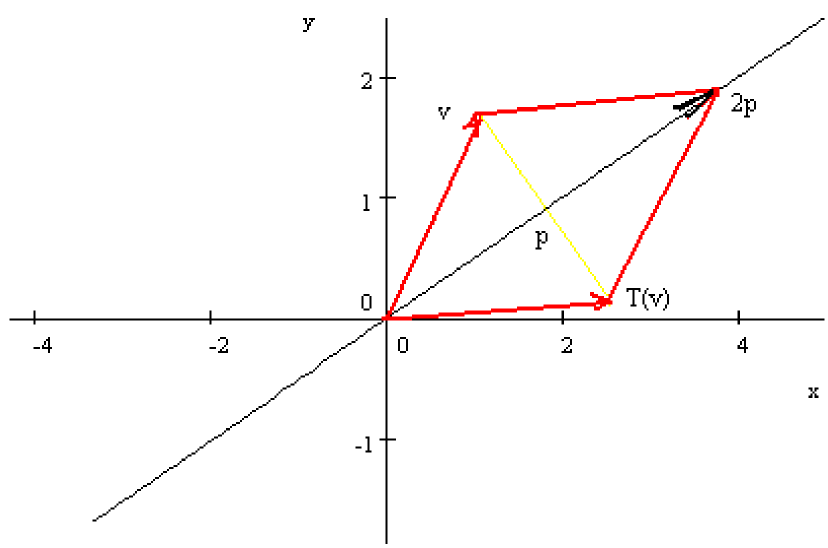
1) Seja $T : \mathbb{R}^2 \to \mathbb{R}^2$ operador que possui os autovetores $u = (1, 0)$ e $v = (1, 1)$ associado aos autovalores $\lambda_1 = 1$ e $\lambda_2 = -1$, respectivamente. Determine $T(x, y)$.

Solução:

$$\begin{aligned} T(1,0) &= 1(1,0) = (1,0) \\ T(1,1) &= -1(1,1) = (-1,-1) \end{aligned}$$

$$\begin{aligned} (x,y) &= a(1,0) + b(1,1) \\ (x,y) &= (x-y)(1,0) + y(1,1) \\ T(x,y) &= (x-y)T(1,0) + yT(1,1) \\ T(x,y) &= (x-y)(1,0) + y(-1,-1) \\ T(x,y) &= (x-2y, y) \end{aligned}$$

2) Seja $T : \mathbb{R}^2 \to \mathbb{R}^2$ o operador linear que é a reflexão em torno da reta $y = \frac{1}{2}x$. Determine $T(x, y, z)$.



Tomando $u=(2,1)$ como vetor diretor da reta temos que

$$\begin{aligned}
T(v)+v &= 2p\\
T(v) &= 2p-v\\
T(v) &= 2proj_u v - v\\
T(x,y) &= 2\left[\frac{(2,1).(x,y)}{(2,1).(2,1)}\right](2,1)-(x,y)\\
T(x,y) &= 2\left(\frac{2x+y}{5}\right)(2,1)-(x,y)\\
T(x,y) &= \left(\frac{8x+4y}{5},\frac{4x+2y}{5}\right)-(x,y)\\
T(x,y) &= \left(\frac{3}{5}x+\frac{4}{5}y,\frac{4}{5}x-\frac{3}{5}y\right)
\end{aligned}$$

3) Mostre que a matriz $A=\begin{bmatrix}1&2\\3&2\end{bmatrix}$ é semelhante à matriz $B=\begin{bmatrix}4&0\\0&-1\end{bmatrix}$

Solução: Devemos encontrar uma matriz P tal que $PAP^{-1}=B$, ou seja, $PA=BP$

$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&2\\3&2\end{bmatrix}=\begin{bmatrix}4&0\\0&-1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$\begin{bmatrix}a+3b&2a+2b\\c+3d&2c+2d\end{bmatrix}=\begin{bmatrix}4a&4b\\-c&-d\end{bmatrix}$

$\begin{cases}-3a+3b=0\\2a-2b=0\end{cases}\Rightarrow b=a$

$\begin{cases}2c+3d=0\\2c+3d=0\end{cases}\Rightarrow c=-d$

Podemos então tomar $P=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\Rightarrow P^{-1}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{bmatrix}$ Logo

$PAP^{-1}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1&2\\3&2\end{bmatrix}\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{bmatrix}=\begin{bmatrix}4&0\\-1&-1\end{bmatrix}=B$

4) Determinar o operador inverso, $T^{-1}(x,y,z)$, do operador dado por $T(x,y,z)=(3x+2z,y,x+z)$

$[T]=\begin{bmatrix}3&0&2\\0&1&0\\1&0&1\end{bmatrix}\Longrightarrow\left[T^{-1}\right]=[T]^{-1}=\begin{bmatrix}1&0&-2\\0&1&0\\-1&0&3\end{bmatrix}$

$$T^{-1}(x,y,z)=(x-2z,y,-x+3z)$$

5) Determinar os autovetores e autovalores da matriz

$$A=\begin{bmatrix}2&0&1\\0&2&0\\2&0&3\end{bmatrix}$$

$\det(A-\lambda I)=\det\begin{bmatrix}2-\lambda&0&1\\0&2-\lambda&0\\2&0&3-\lambda\end{bmatrix}=0$

$\det(A-\lambda I)=(2-\lambda)^2(3-\lambda)-2(2-\lambda)=0$
$\det(A-\lambda I)==(2-\lambda)[(2-\lambda)(3-\lambda)-2]=0$
$\det(A-\lambda I)=(2-\lambda)(\lambda^2-5\lambda+4)=0$

$$\lambda_1=2,\quad \lambda_2=1,\quad \lambda_3=4$$

Para $\lambda_1=2$

$$\begin{bmatrix}0&0&1\\0&0&0\\2&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Longrightarrow \begin{array}{c}z=0\\ \\ x=0\end{array}$$

Associado a $\lambda_1=2$ temos $v_1=(0,y,0)$

Para $\lambda_2=1$

$$\begin{bmatrix}1&0&1\\0&1&0\\2&0&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Longrightarrow \begin{array}{c}-x=z\\ y=0\\ -x=z\end{array}$$

Associado a $\lambda_2=1$ temos $v_2=(-x,0,-x)$

Para $\lambda_3=4$

$$\begin{bmatrix}-2&0&1\\0&-2&0\\2&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Longrightarrow \begin{array}{c}z=2x\\ y=0\\ z=2x\end{array}$$

Associado a $\lambda_3=4$ temos $v_3=(2x,0,2x)$