Série de Leibnitz

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \cdots + \cdots$$

Prova

$\frac{\pi}{4}$ = arc tg (1) = $\int_0^1 \frac{dx}{1+x^2}$

Mas $S_n = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n}$

$\underline{\ x^2 S_n = x^2 - x^4 + x^6 - \ldots + (-1)^n x^{2n+2}}$

$(1+x^2)S_n = 1 + (-1)^n x^{2n+2} \Rightarrow S_n = \frac{1}{1+x^2} + \frac{(-1)^n x^{2n+2}}{1+x^2}$

Assim, $\frac{1}{1+x^2} = S_n - \frac{(-1)^n x^{2n+2}}{1+x^2} = S_n + \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}$

Então $\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2} = \int_0^1 Sn + \frac{(-1)^{n+1} x^{2n+2}}{1+x^2} dx$

Mas $\int_0^1 \frac{(-1)^{n+1} x^{2n+2}}{1+x^2} dx < \int_0^1 x^{2n+2} dx = \frac{1}{2n+3} \to 0$, com n $\to \infty$

Portanto, $\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2} = \int_0^1 Sn\, dx = \int_0^1 \sum_{n-0}^{\infty} (-1)^n x^{2n}\ dx = \sum_{n-0}^{\infty} \frac{(-1)^n}{2n+1}$,

Ou seja $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \ldots - \ldots$ c.q.d.

Também pode ser determinada pela expansão em série de arc tg (x), com $x = 1$

Também (Série de Nilakantha):

$$\pi = 3 + \frac{4}{2.3.4} - \frac{4}{4.5.6} + \frac{4}{6.7.8} - \frac{4}{8.9.10} + \frac{4}{10.11.12} - \frac{4}{12.13.14} + \ldots - \ldots$$

Prova:

$\pi = 3 + S$, onde

$S = \frac{4}{2.3.4} - \frac{4}{4.5.6} + \frac{4}{6.7.8} - \frac{4}{8.9.10} + \frac{4}{10.11.12} - \frac{4}{12.13.14} + \ldots - \ldots$

Ou $S = T_1 - T_2 + T_3 - T_4 + \ldots$

Onde $T_k = \frac{4}{2k(2k+1)(2k+2)}$

Usando frações Parciais. Ficamos com

$T_k = \frac{2}{2k} - \frac{4}{2k+1} + \frac{2}{2k+2}$

e

$T_{k+1} = \frac{2}{2(k+1)} - \frac{4}{2(k+1)+1} + \frac{2}{2(k+1)+2} = \frac{2}{2k+2} - \frac{4}{2k+3} + \frac{2}{2k+4}$

Então,

$T_k - T_{k+1} = \frac{2}{2k} - \frac{4}{2k+1} + \frac{2}{2k+2} - \frac{2}{2k+2} + \frac{4}{2k+3} - \frac{2}{2k+4} = \frac{2}{2k} - \frac{4}{2k+1} + \frac{4}{2k+3} - \frac{2}{2k+4}$

Então, $S_4 = T_1 - T_2 + T_3 - T_4 = \frac{2}{2} - \frac{4}{3} + \frac{4}{5} - \frac{2}{6} + \frac{2}{6} - \frac{4}{7} + \frac{4}{9} - \frac{2}{10}$

Ou seja, $S_4 = \frac{2}{2} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{2}{10}$

Continuando, $S = \frac{2}{2} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \frac{4}{15} + \ldots - \ldots$

Então, $\pi = 3 + S = 3 + \frac{2}{2} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \frac{4}{15} + \ldots - \ldots$

Ou $\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \frac{4}{15} + \ldots - \ldots$

Ou, ainda, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \ldots - \ldots$ (Leibnitz). c.q.d.